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A PLANE NONLINEAR SHEAR FOR AN ELASTIC LAYER WITH A NONCONVEX STORED ENERGY FUNCTION

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1. INTRODUCTION

When the stored energy function for an elastic material is not globally rank-1 convex as a function of the deformation gradient, then equilibrium configurations which possess a discontinuous strain field are possible. In fact, in some applications, a configuration of this sort is determined as one which minimizes the total potential energy of the system, and in such a case the resulting state of the body is said to possess multiple coexistent phases. The strain field in these coexistent phases may be uniform or nonuniform, but in all cases they are separated by surfaces across which the strain itself is discontinuous: Such surfaces are called phase boundaries.

In 1975, Ericksen[1] made the novel observation, within the context of one-dimensional elasticity theory, that a bar in tension can have stable equilibrium states with discontinuous strain fields when the stored energy function is not convex. Shortly thereafter, James[2] magnified upon this idea for elastic bar theory, and almost coincidently Dunn and Fosdick[3] considered analogous questions and relationships between continuum thermodynamics, Gibbsian thermostatics, and the stability of material phases. This latter work† was concerned to some extent with the question of existence, uniqueness and detailed structure of minimizers to various problems from thermostatics, and it introduced the idea of "uniqueness to within rearrangement" and related this idea to the Gibbs phase rule for a single component system.

Since the mid-seventies, there have been many papers written on questions of minimization within the context of mechanics for a material with a nonconvex stored energy function. Specific problems include the pure bending and postbuckling behavior of an elastica[5, 6], the finite twisting of an elastic tube[7] and the helical shear of an elastic tube[8].

In the present work we shall consider another specific and elementary minimization problem in which the nonconvexity of the stored energy function manifests itself as a novel phenomenon. In particular, in this problem an elastic layer is bonded between two infinite parallel rigid plates and the plates are rotated in the same sense through the same angle about *distinct axes* normal to the plates. In general the shear that is thereby induced in the layer is expected to correspond to the relative rotation of planes of material throughout the thickness of the layer and about centers which are dependent upon the position of these planes. In certain situations we find the locus of these centers to be a straight line, and in others we show that it is composed of continuous straight line segments each having one

†A summary of this work can be found in [4].

of two distinct slopes. In this latter case there is considerable nonuniqueness with respect to how the combination of straight line segments may be arranged without affecting the energy of the system. The general structure of all minimizers in this problem is completely determined.

2. SETTING THE PROBLEM

Consider a plane layer of elastic, incompressible, isotropic and homogeneous material B of thickness $2h$ which is bonded between two infinite parallel rigid plates, and identify B with the region of space defined by the rectangular Cartesian coordinates $x_1 \in (-\infty, \infty)$, $x_2 \in (-\infty, \infty)$, $x_3 \in (-h, h)$. The layer is said to undergo a *staggered axis twist* if the deformation $(x_i) \to (y_i)$ measured relative to a fixed coordinate system has the form (cf. Rajagopal and Wineman[9])

$$
\begin{aligned}
y_1 &= (x_1 - f_1(x_3)) \cos \Omega - (x_2 - f_2(x_3)) \sin \Omega + f_1(x_3), \\
y_2 &= (x_1 - f_1(x_3)) \sin \Omega + (x_2 - f_2(x_3)) \cos \Omega + f_2(x_3), \\
y_3 &= x_3 \equiv z,
\end{aligned}
\tag{1}
$$

where, it should be noted, we have introduced in $(1)_3$ the more convenient notation z for x_3. This deformation has the property that every plane $z =$ constant is rotated through the same constant prescribed angle $\Omega \in [0, 2\pi)$ about an axis parallel to the z-coordinate line, which has its center of rotation located at the point $(f_1(z) f_2(z), z)$. The locus of these centers of rotation for $z \in (-h, h)$ generates a spatial curve which, if known, would completely define the deformation.

In a recent paper, Rajagopal and Wineman[9] considered the possibility of determining those continuous functions f_1 and f_2 which lead to equilibrated states, and they found that deformation fields wherein the derivatives f_1' and f_2' are discontinuous were possible. Their work was not concerned with questions of minimization or stability, but rather was centered on the notion of equilibrium. Two elementary facts, observed in [9], are relevant here: First, the deformation is isochoric for all forms of f_1 and f_2, and second, the principal invariants of either the right or left Cauchy–Green strain tensor are given by

$$
I = II = 3 + [\kappa(z)]^2, \qquad III = 1, \tag{2}
$$

where the shear $\kappa(z)$ is defined by

$$
\kappa(z) \equiv 2[[f_1'(z)]^2 + [f_2'(z)]^2]^{1/2} \sin(\Omega/2) \geq 0. \tag{3}
$$

It is convenient in this work to introduce the pair of strains $\boldsymbol{\varepsilon} \equiv (\varepsilon_1, \varepsilon_2)$, where

$$
\varepsilon_\alpha(z) \equiv 2 f_\alpha'(z) \sin(\Omega/2), \qquad \alpha = 1, 2, \tag{4}
$$

so that (3) may be written as

$$
\kappa = |\boldsymbol{\varepsilon}|.
$$

In this case, the stored energy density per unit volume of the material has the following forms:

$$
\begin{aligned}
W = W(I, II) &= W(3 + \kappa^2, 3 + \kappa^2) \\
&= \tilde{W}(\kappa) \\
&= \bar{W}(\boldsymbol{\varepsilon}).
\end{aligned}
\tag{5}
$$

We shall suppose that $\tilde{W}:[0,\beta)\to\mathbb{R}$, where $0<\beta\le\infty$, so that $\hat{W}:\mathscr{D}\to\mathbb{R}$, where $\mathscr{D}\equiv\{\boldsymbol{\varepsilon}\in\mathbb{R}_2\,|\,|\boldsymbol{\varepsilon}|<\beta\}$. Clearly, $\hat{W}(-\boldsymbol{\varepsilon})=\hat{W}(\boldsymbol{\varepsilon})$.

The shear stress is given by

$$\tau(\kappa)\equiv\frac{\mathrm{d}\tilde{W}(\kappa)}{\mathrm{d}\kappa}\equiv\tilde{W}'(\kappa), \tag{6}$$

and it follows that

$$\tau(\kappa)=\kappa\mu(\kappa), \tag{7}$$

where

$$\mu(\kappa)=2\left\{\frac{\partial W(I,II)}{\partial I}+\frac{\partial W(I,II)}{\partial II}\right\}\bigg|_{I=II=3+\kappa^2}. \tag{8}$$

In addition, we also have

$$\hat{W}_{\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon})=\mu(\kappa)\boldsymbol{\varepsilon}, \tag{9}$$

where the subscript $\boldsymbol{\varepsilon}$ denotes the gradient operation.

In this work we shall require $\tilde{W}$ and τ to behave in the manner as depicted in Figs. 1 and 2. More precisely, we shall impose:

(i) *τ is class C^2.*

(ii) *There exists κ_1 and κ_2 with $0<\kappa_1<\kappa_2<\beta$ such that*

$$\begin{aligned}&\text{(a)}\quad \tau'(\kappa_1)=\tau'(\kappa_2)=0,\\ &\text{(b)}\quad \tau'(\kappa)>0\quad \forall\kappa\in[0,\kappa_1)\cup(\kappa_2,\beta),\\ &\qquad\ \ \tau'(\kappa)<0\quad \forall\kappa\in(\kappa_1,\kappa_2).\end{aligned}$$

(iii) *There exists κ_3 and κ_4 with $0<\kappa_3<\kappa_1<\kappa_2<\kappa_4<\beta$ such that*

$$\begin{aligned}&\text{(a)}\quad \tau(\kappa_3)=\tau(\kappa_4)\equiv\tau_c,\\ &\text{(b)}\quad \tilde{W}(\kappa_4)-\tilde{W}(\kappa_3)=\tau_c(\kappa_4-\kappa_3).\end{aligned}$$

In words, the above three conditions require $\tilde{W}$ to be sufficiently smooth, convex in the

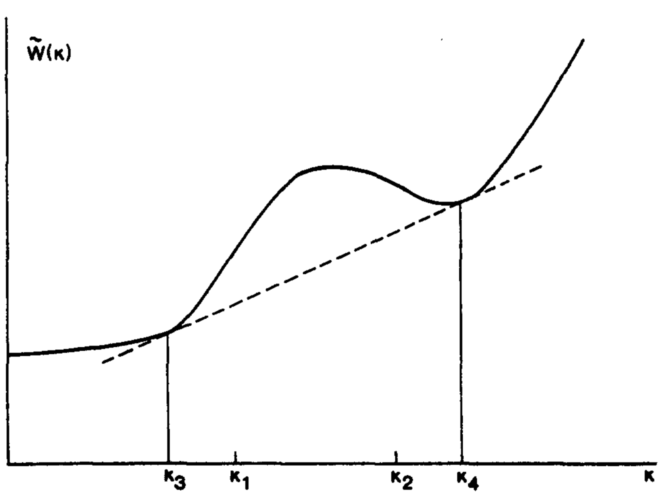

Fig. 1. The stored energy as a function of shear.

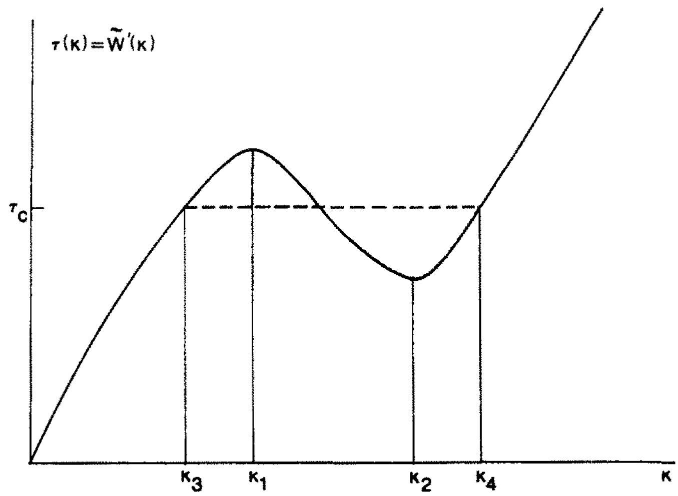

Fig. 2. The shear stress as a function of shear.

domain $[0, \kappa_3] \cup [\kappa_4, \beta)$ and not convex over the interval (κ_3, κ_4). The values κ_1 and κ_2 correspond to points in the interval (κ_3, κ_4) where local convexity breaks down. The points κ_3 and κ_4 are the extremities of the so-called *Maxwell equal area line.*

In the remainder of this work we shall consider an energy method to determine those deformations of the form (1) for which $f_1(z)$ and $f_2(z)$ are prescribed at $z = \pm h$. For this purpose, then, we first introduce the total potential energy functional for the elastic layer per unit cross-sectional area, i.e.

$$\mathscr{E}[\boldsymbol{\varepsilon}] = \int_{-h}^{h} \tilde{W}(\kappa(z))\,\mathrm{d}z = \int_{-h}^{h} \bar{W}(\boldsymbol{\varepsilon}(z))\,\mathrm{d}z, \tag{10}$$

where κ and $\boldsymbol{\varepsilon}$ are given in (3) and (4). Also, it is helpful to observe that because of (4), we have

$$\int_{-h}^{h} \boldsymbol{\varepsilon}(z)\,\mathrm{d}z = 2[\mathbf{f}(h) - \mathbf{f}(-h)]\sin(\Omega/2) \equiv \boldsymbol{\Delta}, \tag{11}$$

where $\boldsymbol{\Delta}$ is a prescribed pair of numbers. Finally, then, the minimization problem which we shall consider is

$$\underset{\boldsymbol{\varepsilon}\in\mathscr{A}}{\text{minimize}}\,\mathscr{E}[\boldsymbol{\varepsilon}], \tag{12}$$

where the class $\mathscr{A}$ of admissible strains is given by

$$\mathscr{A} \equiv \left\{\boldsymbol{\varepsilon} : (-h, h) \to \mathscr{D} \,|\, \boldsymbol{\varepsilon} \textit{ is piecewise continuous}, \int_{-h}^{h} \boldsymbol{\varepsilon}(z)\,\mathrm{d}z = \boldsymbol{\Delta}\right\}.$$

3. ANALYSIS OF THE VARIATIONAL PROBLEM; STRUCTURE OF SOLUTIONS

The variational problem (10), (12) is of a standard type and, thus, we shall simply record, without proof, the usual *necessary* conditions of Euler–Lagrange and Weierstrass for the existence of a minimizer. We shall then show that these necessary conditions are also sufficient for existence by constructing all possible solutions.

Suppose throughout this section that $\boldsymbol{\varepsilon} \in \mathscr{A}$ is a minimizer of (10), (12). Then, on subintervals of $(-h, h)$ over which ε is continuous, the Euler–Lagrange equation has the form

$$\mu(\kappa(z))\varepsilon(z) = \mathbf{c}, \tag{13}$$

where $\mathbf{c} = (c_1, c_2)$ is constant. Moreover, as a further consequence of the first variation condition it can be shown that the left-hand side of (13) has zero jump at points where ε is discontinuous. Thus, *the constant* $\mathbf{c}$ *is the same for all* $z \in (-h, h)$, and if one is seeking a minimizer, it must necessarily solve the Euler–Lagrange boundary value problem which consists in identifying any member of $\mathscr{A}$ which satisfies (13).

The necessary condition of Weierstrass for the problem (10), (12) has the form

$$\hat{W}(\varepsilon_1) \geq \hat{W}(\varepsilon(z)) + \hat{W}_\varepsilon(\varepsilon(z)) \cdot [\varepsilon_1 - \varepsilon(z)], \tag{14}$$

which is to hold for all points $z \in (-h, h)$ at which $\varepsilon(z)$ is continuous, and for all $\varepsilon_1 \in \mathscr{D}$. Thus, a minimizer ε must be such that its values $\varepsilon(z)$ at any point of continuity $z \in (-h, h)$ is also a point in $\mathscr{D}$ at which $\hat{W}$ is convex. *Equivalently*, by Lemma 1.1 of Fosdick and MacSithigh[8], it follows that at any such point of continuity, $\kappa(z) = |\varepsilon(z)|$ must be associated with a point of convexity of $\tilde{W}$ in $[0, \beta)$ at which $\tilde{W}'(\kappa(z))$ is nonnegative, i.e.

$$\tilde{W}(\kappa_1) \geq \tilde{W}(\kappa(z)) + \tilde{W}'(\kappa(z))[\kappa_1 - \kappa(z)], \qquad \tilde{W}'(\kappa(z)) \geq 0, \tag{15}$$

for all $\kappa_1 \in [0, \beta)$. From Fig. 1 we see that this requires that the range of the shear κ for any minimizer is restricted to the set $[0, \kappa_3] \cup [\kappa_4, \beta)$.

We now have the minimization

THEOREM. *Suppose* $\varepsilon \in \mathscr{A}$ *is a solution of* (13) *which satisfies* (14) (*or equivalently* (15)). *Then, among all other* $\varepsilon^* \in \mathscr{A}$,

$$\mathscr{E}[\varepsilon^*] \geq \mathscr{E}[\varepsilon]. \tag{16}$$

Proof. From (9) and (13) we see that

$$\hat{W}_\varepsilon(\varepsilon(z)) = \mu(\kappa(z))\varepsilon(z) = \mathbf{c},$$

and this, in turn, yields

$$\int_{-h}^{h} \hat{W}_\varepsilon(\varepsilon(z)) \cdot [\varepsilon^*(z) - \varepsilon(z)]\,\mathrm{d}z = \mathbf{c} \cdot \int_{-h}^{h} [\varepsilon^*(z) - \varepsilon(z)]\,\mathrm{d}z = 0,$$

the latter equality following from the fact that both ε and ε^* are members of the class $\mathscr{A}$. Thus, we readily conclude that

$$\mathscr{E}[\varepsilon^*] - \mathscr{E}[\varepsilon] = \int_{-h}^{h} [\hat{W}(\varepsilon^*(z)) - \hat{W}(\varepsilon(z))]\,\mathrm{d}z$$

$$= \int_{-h}^{h} [\hat{W}(\varepsilon^*(z)) - \hat{W}(\varepsilon(z)) - \hat{W}_\varepsilon(\varepsilon(z)) \cdot [\varepsilon^*(z) - \varepsilon(z)]]\,\mathrm{d}z,$$

which by (14), yields (16) to complete the proof.

The above theorem does not directly address the question of existence or uniqueness of minimizers. It is certain, however, from this theorem that all members of the set of admissible functions $\mathscr{A}$ that solve (13) and satisfy (14) (or (15)) must have the same minimum value of $\mathscr{E}$. We now turn to the existence and uniqueness of such minimizers. In fact, we shall identify the detailed structure of all minimizing fields in the class $\mathscr{A}$.

We begin with the Euler–Lagrange equation (13), and observe that from it we obtain

$$\mu(\kappa(z))\,|\varepsilon(z)| = |\mathbf{c}|, \qquad z \in (-h, h), \tag{17}$$

which with (7) implies

$$\tau(\kappa(z)) = |\mathbf{c}|, \qquad z \in (-h, h). \tag{18}$$

In addition, by substituting (17) back into (13) we reach

$$\boldsymbol{\varepsilon}(z) = \frac{\mathbf{c}}{|\mathbf{c}|}\kappa(z), \tag{19}$$

and since $\boldsymbol{\varepsilon} \in \mathscr{A}$, we see that

$$\int_{-h}^{h} \boldsymbol{\varepsilon}(z)\,\mathrm{d}z = \boldsymbol{\Delta} = \frac{\mathbf{c}}{|\mathbf{c}|}\int_{-h}^{h} \kappa(z)\,\mathrm{d}z, \tag{20}$$

and

$$|\boldsymbol{\Delta}| = \int_{-h}^{h} \kappa(z)\,\mathrm{d}z. \tag{21}$$

From this, we see immediately that *a minimizer in $\mathscr{A}$ does not exist if $\boldsymbol{\Delta}$ is such that* $|\boldsymbol{\Delta}|/2h \geq \beta$. This follows since the range of values accessible to $\kappa(z)$, for $z \in (-h, h)$, is the half-open interval $[0, \beta)$, and (21) requires that its mean value be $|\boldsymbol{\Delta}|/2h$.

Actually, as remarked earlier, the range of values accessible to the shear $\kappa(z)$ for any minimizer, for $z \in (-h, h)$, is the set $[0, \kappa_3] \cup [\kappa_4, \beta)$, and this fact will be an important issue in the remaining arguments. In fact, since (21) requires that the number $|\boldsymbol{\Delta}|/2h$ be the mean value of any minimizing shear field $\kappa(z)$, $z \in (-h, h)$, and because of the above restriction on the range of $\kappa(z)$ for $z \in (-h, h)$, and the fact that the shear stress associated with any minimizer, $\tau(\kappa(z))$, must be constant as indicated in (18), it readily follows, basically from Fig. 2, that

$$\frac{|\boldsymbol{\Delta}|}{2h} \in [0, \kappa_3] \cup [\kappa_4, \beta) \Rightarrow \kappa(z) = \frac{|\boldsymbol{\Delta}|}{2h}, \qquad z \in (-h, h). \tag{22}$$

By use of (19) and (20) we then see that *for any $\boldsymbol{\Delta}$ such that $|\boldsymbol{\Delta}|/2h \in [0, \kappa_3] \cup [\kappa_4, \beta)$ we have (uniquely)*

$$\boldsymbol{\varepsilon}(z) = \frac{\boldsymbol{\Delta}}{2h}, \qquad z \in (-h, h). \tag{23}$$

Because of (4) we see that in this case the centers of rotation for the deformation (1) all lie on a common straight line which connects the two points $(f_1(\pm h), f_2(\pm h), \pm h)$.

Now, let us suppose that $|\boldsymbol{\Delta}|/2h \in (\kappa_3, \kappa_4)$. Again $|\boldsymbol{\Delta}|/2h$ must be the mean value of any minimizing shear field $\kappa(z)$, $z \in (-h, h)$, and the shear stress $\tau(\kappa(z))$ must be constant throughout this interval. Since the only range of values open to $\kappa(z)$, $z \in (-h, h)$, is $[0, \kappa_3] \cup [\kappa_4, \beta)$, and since κ_3 and κ_4 are the only two separate points in this set having common shear stress values, it follows that

$$\kappa(z) = \begin{cases} \kappa_3, & z \in P, \\ \kappa_4, & z \in (-h, h)\backslash P, \end{cases} \tag{24}$$

where the set P is a finite union of subintervals of $(-h, h)$ which must have a total length, meas(P), that is compatible with (21), i.e.

$$|\Delta| = \kappa_3 \operatorname{meas}(P) + \kappa_4(2h - \operatorname{meas}(P)).$$

Thus,

$$\operatorname{meas}(P) = \left(\frac{\kappa_4 - |\Delta|/2h}{\kappa_4 - \kappa_3}\right) 2h. \tag{25}$$

By use of (19) and (20) we then see that *for any* Δ *such that* $|\Delta|/2h \in (\kappa_3, \kappa_4)$ *we have*

$$\varepsilon(z) = \begin{cases} \dfrac{\Delta}{|\Delta|}\kappa_3, & z \in P, \\ \dfrac{\Delta}{|\Delta|}\kappa_4, & z \in (-h, h)\backslash P. \end{cases} \tag{26}$$

In this case, we see from (1) and (4) that the centers of rotation of this deformation may lie on *any* arrangement of continuous piecewise straight line segments which connect the two points $(f_1(\pm h), f_2(\pm h), \pm h)$, and which have the two distinct slopes compatible with (4), (25) and (26). Because only the total length meas(P) is determined, the set of minimizers here is said to be *unique up to a rearrangement.*

We conclude this paper with the following brief comment concerning the above non-uniqueness of solution. The question of uniqueness up to a rearrangement seems to arise only in minimum problems associated with homogeneously deformed homogeneous bodies. Indeed, examples of inhomogeneous *unique* minimizers are known (e.g. [7, 8]), and others can be generated, even when phase boundaries are present.

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